

A note on stochastic calculus in vector bundles

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Abstract

The aim of these notes is to relate covariant stochastic integration in a vector bundle E (as in Norris [6]) with the usual Stratonovich calculus via the connector $\mathcal{K}_\nabla : TE \rightarrow E$ (cf. e.g. Paterson [7] or Poor [8]) which carries the connection dependence.

Key words and phrases: Vector bundles, global analysis, stochastic calculus.

AMS 2010 subject classification: 58J65 (60J60, 60H05).

1 Introduction

Stochastic calculus on vector bundles has been studied by several authors, among others, Arnaudon and Thalmaier [1], Norris [6], Driver and Thalmaier [3]. In these articles, the stochastic integral of a semimartingale v_t in a vector bundle $\pi : E \rightarrow M$ is defined by decomposing v_t into horizontal and vertical (covariant) components according to a given connection in E . The aim of these notes is to relate covariant stochastic integral in vector bundles (Norris [6]) with the usual Stratonovich calculus using an appropriate operator, the connector \mathcal{K}_∇ (cf. e.g. Paterson [7] and Poor [8]), from the tangent space TE to E which carries the connection dependence.

We denote by M a smooth differentiable manifold. Let E be an n -dimensional vector bundle over M endowed with a connection ∇ . This connection induces a natural projection $\mathcal{K}_\nabla : TE \rightarrow E$ called the associated connector (cf. Paterson [7] and Poor [8]) which projects into the vertical subspace of TE identified with E . More precisely: Given a differentiable curve $v_t \in E$, decompose $v_t = u_t f_t$, where u_t is the

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unique horizontal lift of $\pi(v_t)$ in the principal bundle $Gl(E)$ of frames in E starting at a certain u_0 with $\pi(u_0) = \pi(v_0)$ and $f_t \in \mathbf{R}^n$. Then

$$\mathcal{K}_\nabla(v'_0) := u_0 f'_0.$$

Norris [6] defines the covariant Stratonovich integration of a section θ in the dual vector bundle E^* along a process $v_t \in E$ by:

$$\int \theta Dv_t := \int \theta u_t \circ df_t,$$

where $v_t = u_t f_t$; and the corresponding covariant Itô version:

$$\int \theta D^I v_t := \int \theta u_t df_t.$$

2 Main results

Initially, observe that using the connector \mathcal{K}_∇ , the covariant integral above reduces to a classical Stratonovich integral of 1-forms:

Proposition 2.1 *Let v_t be a semimartingale in E and $\theta \in \Gamma(E^*)$. Then*

$$\int \theta Dv_t = \int \theta \mathcal{K}_\nabla \circ dv_t.$$

Proof: Let $\phi : Gl(E) \times \mathbf{R}^n \rightarrow E$ be the action map $\phi(u, f) = uf$. The right hand side in the equation above is

$$\int \theta \mathcal{K}_\nabla \circ d\phi(u_t, f_t) = \int \phi_{u_t}^* \theta \mathcal{K}_\nabla \circ df_t + \int \phi_{f_t}^* \theta \mathcal{K}_\nabla \circ du_t. \quad (1)$$

The second term on the right hand side vanishes since $\phi_{f_t}^* \theta \mathcal{K}_\nabla = 0$. The formula holds because $\phi_{u_t}^* \mathcal{K}_\nabla(z) = u_t z$ for all $z \in \mathbf{R}^n$. □

Remark: In the special case of $E = TM$, one can compare the classical integration of 1-forms with the covariant integration: Let Y_t be a semimartingale in M and v_t be a semimartingale in E . If $v_t = u_t f_t$ such that u_t is a horizontal lift of Y_t and f_t is the antidevelopment of Y_t , then for any 1-form θ , the classical integration in M and the covariant integration in E coincides:

$$\int \theta \circ dY_t = \int \theta \mathcal{K}_\nabla \circ dv_t.$$

Local coordinates:

Let $\{\delta_1, \dots, \delta_n\}$ be local sections in E which is a basis in a coordinate neighbourhood $(U, \varphi = (x^1, \dots, x^d))$, where d is the dimension of M . For $1 \leq \alpha, \beta \leq n$ and $1 \leq i \leq d$, we write

$$\nabla_{\frac{\partial}{\partial x^i}} \delta_\alpha = \Gamma_{i\alpha}^\beta \delta_\beta,$$

then

$$\mathcal{K}_\nabla \left(\frac{\partial \delta_\alpha}{\partial x^i} \right) = \Gamma_{i\alpha}^\beta \delta_\beta.$$

Let γ_t be a differentiable curve in M and u_t be a horizontal lift of γ_t in $Gl(E)$, we write $u_t^\beta = u_t(e_\beta) = u_t^{\beta\alpha} \delta_\alpha(\gamma_t)$. Naturally

$$\nabla_{\gamma'_t} u_t^\beta = 0,$$

and the parallel transport equation is given by

$$\frac{u_t^{\alpha\beta}}{dt} + \frac{\gamma^j}{dt} u_t^{\alpha\gamma} \Gamma_{j\gamma}^\beta(\gamma_t) = 0.$$

For $\theta \in \Gamma(E^*)$, write $\theta = \theta^\alpha \delta_\alpha^*$, where $\theta^\alpha = \theta(\delta_\alpha)$. We have, for each $1 \leq \alpha \leq n$

$$\left(\nabla_{\frac{\partial}{\partial x^j}} \theta \right) \delta_\alpha = \frac{\partial \theta^\alpha}{\partial x^j} - \theta \left(\nabla_{\frac{\partial}{\partial x^j}} \delta_\alpha \right) = \frac{\partial \theta^\alpha}{\partial x^j} - \Gamma_{j\alpha}^\beta \theta^\beta.$$

That is,

$$\nabla \theta = \left(\frac{\partial \theta^\alpha}{\partial x^j} - \Gamma_{j\alpha}^\beta \theta^\beta \right) dx^j \otimes \delta_\alpha^*.$$

Cross quadratic variation in sections of $TM^* \otimes E^*$:

In order to find a covariant conversion formula for Itô-Stratonovich integrals we introduce stochastic integration formulae for sections of $TM^* \otimes E^*$, which is the space where the covariant derivative $\nabla \theta$ lives. Let v_t be a semimartingale in E . Denoting $x_t = \pi(v_t)$, we have the following identities:

1) For $\alpha \in \Gamma(TM^*)$ and $\theta \in \Gamma(E^*)$,

$$\int \alpha \otimes \theta (dx_t, Dv_t) = \left\langle \int \alpha \circ d\pi(v_t), \int \theta Dv_t \right\rangle.$$

2) For $b \in \Gamma(TM^* \otimes E^*)$ and $f \in C^\infty(M)$,

$$\int f b (dx_t, Dv_t) = \int f(\pi(v_t)) \circ d \int b (dx_t, Dv_t).$$

This is well defined (similarly to Emery [5, p. 23]). In particular, for $b = \nabla\theta$, in local coordinates:

$$\begin{aligned}\int \nabla\theta (dx_t, Dv_t) &= \int \left(\frac{\partial\theta^\alpha}{\partial x^j} - \Gamma_{j\alpha}^\beta \theta^\beta \right) \circ d \int dx^j \otimes \delta_\alpha^* (dx_t, Dv_t) \\ &= \int \left(\frac{\partial\theta^\alpha}{\partial x^j} - \Gamma_{j\alpha}^\beta \theta^\beta \right)(x_t) \circ d < x_t^j, \int u^{\gamma\alpha} df^\gamma >_t \\ &= \int \left(\frac{\partial\theta^\alpha}{\partial x^j} - \Gamma_{j\alpha}^\beta \theta^\beta \right)(x_t) u_t^{\gamma\alpha} \circ d < x^j, f^\gamma >_t.\end{aligned}$$

We have the following Itô-Stratonovich covariant conversion formula:

Proposition 2.2 *Let v_t be a semimartingale in E and $\theta \in \Gamma(E^*)$. Then*

$$\int \theta Dv_t = \int \theta D^I v_t + \frac{1}{2} \int \nabla\theta (dx_t, Dv_t). \quad (2)$$

Proof: In local coordinates we have that

$$\int \theta Dv_t = \int \theta_{x_t}(u_t e_\alpha) \circ df_t^\alpha = \int \theta_{x_t}(u_t e_\alpha) df_t^\alpha + \frac{1}{2} \langle \theta(ue_\alpha), f^\alpha \rangle.$$

We have to show that

$$\int \nabla\theta (dx_t, Dv_t) = \langle \theta(ue_\alpha), f^\alpha \rangle.$$

But

$$\begin{aligned}\langle \theta(ue_\alpha), f^\alpha \rangle &= \langle \theta_x^\beta \delta_\beta^*(ue_\alpha), f^\alpha \rangle \\ &= \langle \theta_x^\beta (ue_\alpha)^\beta, f^\alpha \rangle \\ &= \int (ue_\alpha)^\beta d \langle \theta_x^\beta, f^\alpha \rangle + \int \theta_x^\beta d \langle (ue_\alpha)^\beta, f^\alpha \rangle \\ &= \int u^{\alpha\beta} d \langle \theta_x^\beta, f^\alpha \rangle + \int \theta_x^\beta d \langle u^{\alpha\beta}, f^\alpha \rangle \\ &= \int u^{\alpha\beta} \frac{\partial\theta^\beta}{\partial x^j} d \langle x^j, f^\alpha \rangle + \int \theta_x^\beta d \langle u^{\alpha\beta}, f^\alpha \rangle \\ &= \int u^{\alpha\beta} \frac{\partial\theta^\beta}{\partial x^j} d \langle x^j, f^\alpha \rangle - \int \theta_x^\beta u^{\alpha\gamma} \Gamma_{j\gamma}^\beta(x) \langle x^j, f^\alpha \rangle \\ &= \int \left(u^{\alpha\beta} \frac{\partial\theta^\beta}{\partial x^j} - \theta_x^\beta u^{\alpha\gamma} \Gamma_{j\gamma}^\beta(x) \right) \langle x^j, f^\alpha \rangle \\ &= \int \left(u^{\alpha\gamma} \frac{\partial\theta^\gamma}{\partial x^j} - \theta_x^\beta u^{\alpha\gamma} \Gamma_{j\gamma}^\beta(x) \right) \langle x^j, f^\alpha \rangle \\ &= \int \left(\frac{\partial\theta^\gamma}{\partial x^j} - \theta_x^\beta \Gamma_{j\gamma}^\beta(x) \right) u^{\alpha\gamma} \langle x^j, f^\alpha \rangle \\ &= \int \nabla\theta (dx_t, Dv_t).\end{aligned}$$

□

Itô representation:

The vertical lift of an element $w \in E$ to the tangent space $T_e E$, with e and w in the same fiber is given by

$$w^v = \frac{d}{dt}[e + tw]_{t=0} \in T_e E. \quad (3)$$

Let r, s be sections of E and X, Y be vector fields of M . We shall consider a connection ∇^h in E , a prolongation of ∇ , which satisfies the following:

$$\begin{aligned} \nabla_{r^v}^h s^v &= 0, & \nabla_{X^h}^h s^v &= (\nabla_X s)^v, \\ \nabla_{r^v}^h Y^h &= 0, & \nabla_{X^h}^h Y^h &\text{ is horizontal.} \end{aligned}$$

Remark: An example of this connections is the horizontal connection defined by Arnaudon and Thalmaier [1], where, considering a connection $\tilde{\nabla}$ in M , the extra condition $\nabla_{X^h}^h Y^h = (\tilde{\nabla}_X Y)^h$ characterizes this connection.

Next proposition shows a geometrical characterization of the covariant Itô integral.

Proposition 2.3 *Let v_t be a semimartingale in E and $\theta \in \Gamma(E^*)$. Then*

$$\int \theta D^I v_t = \int \theta \mathcal{K}_{\nabla} d^{\nabla^h} v_t. \quad (4)$$

Proof: We have to calculate each component of $\nabla^h \theta \mathcal{K}_{\nabla}$. Using that for A, B vector field in E we have that

$$\nabla_A^h \theta \mathcal{K}_{\nabla}(B) = A(\theta \mathcal{K}_{\nabla}(B)) - \theta \mathcal{K}_{\nabla}(\nabla_A^h B),$$

we obtain the components

$$\begin{aligned} \nabla_{r^v}^h \theta \mathcal{K}_{\nabla}(s^v) &= 0, & \nabla_{r^v}^h \theta \mathcal{K}_{\nabla}(Y^h) &= 0, \\ \nabla_{X^h}^h \theta \mathcal{K}_{\nabla}(s^v) &= \nabla_X \theta(s) \circ \pi, & \nabla_{X^h}^h \theta \mathcal{K}_{\nabla}(Y^h) &= 0. \end{aligned}$$

Hence, using Itô- Stratonovich conversion formula for classical 1-form integration, see e.g. Catuogno and Stelmastchuk [2]:

$$\begin{aligned} \int \theta Dv_t &= \int \theta \mathcal{K}_{\nabla} \circ dv_t \\ &= \int \theta \mathcal{K}_{\nabla} d^{\nabla^h} v_t + \frac{1}{2} \int \nabla^h \theta \mathcal{K}_{\nabla}(dv_t, dv_t). \end{aligned}$$

For the correction term, we have that:

$$\nabla^h \theta \mathcal{K}_{\nabla} = \nabla \theta (\pi_* \times \mathcal{K}_{\nabla}),$$

in the sense that $\nabla_{\pi_* A}^h \theta \mathcal{K}_\nabla(B) = \nabla \theta (\pi_* \times \mathcal{K}_\nabla)(A, B)$. But

$$\int \nabla^h \theta \mathcal{K}_\nabla(dv_t, dv_t) = \int \nabla \theta(dx_t, Dv_t).$$

Combining with equation (2), we have that

$$\int \theta D^I v_t = \int \theta \mathcal{K}_\nabla d^{\nabla^h} v_t.$$

□

Vector bundle mappings

Consider two vector bundles $\pi : E \rightarrow M$, $\pi' : E' \rightarrow M'$ and a differentiable fibre preserving mapping $F : E \rightarrow E'$ over a differentiable map $\tilde{F} : M \rightarrow M'$, i.e. $\pi' \circ F = \tilde{F} \circ \pi$.

Let \mathcal{K}_∇ and \mathcal{K}'_∇ be connectors in E and in E' respectively. We define the vertical derivative (or derivative in the fibre) of F in the direction of w by:

$$D^v F(e)(w) = \mathcal{K}'_\nabla F_*(w^v),$$

where the vertical component w^v is given by Equation (3). For $Z \in T_{\pi(e)}M$, the horizontal (or parallel) derivative is:

$$D^h F(e)(Z) = \mathcal{K}'_\nabla F_*(Z^h).$$

For a vector field X in E , we have that

$$X = (\pi_* X)^h + \mathcal{K}_\nabla(X),$$

hence

$$\mathcal{K}'_\nabla F_*(X) = D^v F(\mathcal{K}_\nabla(X)) + D^h F(\pi_*(X)). \quad (5)$$

The Itô formula for the Stratonovich covariant integration includes an usual 1-form integration, compare with Norris [6, Eq. (20)]:

Proposition 2.4 *Given a fibre preserving map F as above,*

$$\int \theta DF(v_t) = \int (D^v F)^* \theta Dv_t + \int (D^h F)^* \theta \circ d(\pi v_t). \quad (6)$$

Proof: We just have to use the decomposition of Equation (5).

$$\begin{aligned} \int \theta DF(v_t) &= \int \theta \mathcal{K}'_\nabla F_* \circ dv_t \\ &= \int (\theta D^v F \mathcal{K}_\nabla + \theta D^h F \pi_*) \circ dv_t \\ &= \int (D^v F)^* \theta Dv_t + \int (D^h F)^* \theta \circ d(\pi v_t). \end{aligned}$$

□

Proposition 2.5 *For a section b' in $(TM')^* \otimes (E')^*$ and a fibre preserving map $F : E \rightarrow E'$ over $\tilde{F} : M \rightarrow M'$ we have that*

$$\int b'(d\pi' F(v_t), DF(v_t)) = \int (\tilde{F}_* \otimes D^v F)^* b'(d\pi v_t, Dv_t) + \int (\tilde{F}_* \otimes D^h F)^* b'(d\pi v_t, d\pi v_t).$$

Proof: We have

$$\begin{aligned} \int b'(d\pi' F(v_t), DF(v_t)) &= \int b'(\pi'_* \otimes \mathcal{K}'_\nabla)(dF(v_t), dF(v_t)) \\ &= \int b'(\pi'_* \otimes \mathcal{K}'_\nabla)(F_* \otimes F_*)(dv_t, dv_t). \end{aligned}$$

Using that

$$(\pi'_* \otimes \mathcal{K}'_\nabla)(F_* \otimes F_*) = \tilde{F}_* \pi_* \otimes (D^v F \mathcal{K}_\nabla + D^h F \pi_*)$$

yields

$$\int b'(d\pi' F(v_t), DF(v_t)) = \int (\tilde{F}_* \otimes D^h F)^* b'(d\pi v_t, Dv_t) + \int (\tilde{F}_* \otimes D^h F)^* b'(d\pi v_t, d\pi v_t).$$

□

Itô version of Formula (6) is given by:

Proposition 2.6 *Given a fibre preserving map F as above,*

$$\begin{aligned} \int \theta D^I F(v_t) &= \int (D^v F)^* \theta D^I v_t + \int (D^h F)^* \theta \circ d\pi v_t + \\ &\quad \frac{1}{2} \int \left(\nabla(D^v F^* \theta) - (\tilde{F}_* \otimes D^V F)^* \nabla' \theta \right) (d\pi v_t, Dv_t) + \\ &\quad \frac{1}{2} \int \left(\tilde{F}_* \otimes D^h F \right)^* \nabla' \theta(d\pi v_t, d\pi v_t). \end{aligned}$$

Proof: By Proposition 2.2 we have that

$$\int \theta D^I F(v_t) = \int \theta DF(v_t) - \frac{1}{2} \int \nabla' \theta(d\pi' F(v_t), DF(v_t))$$

and

$$\int (D^v F)^* \theta Dv_t = \int (D^v F)^* \theta D^I v_t + \frac{1}{2} \nabla(D^v F)^*(d\pi v_t, Dv_t).$$

But, Proposition 2.4 says that:

$$\int \theta DF(v_t) = \int (D^v F)^* \theta Dv_t + \int (D^h F)^* \theta \circ d(\pi v_t).$$

Finally, by Proposition 2.5, we have that

$$\begin{aligned} \int \nabla' \theta(d\pi' F(v_t), DF(v_t)) &= \int (\tilde{F}_* \otimes D^v F)^* \nabla' \theta(d\pi v_t, Dv_t) + \\ &\quad \int (\tilde{F}_* \otimes D^h F)^* \nabla' \theta(d\pi v_t, d\pi v_t), \end{aligned}$$

which implies the formula. □

3 Applications

Commutation Formulae

Given a differentiable map $(a, b) \in \mathbf{R}^2 \mapsto E$, let $s_E : TTE \rightarrow TTE$ be the symmetry map given by $s_E(\partial_a \partial_b s(a, b)) = \partial_b \partial_a s(a, b)$. Let $C = \mathcal{K} \mathcal{K}_* - \mathcal{K} \mathcal{K}_* s_E : TTE \rightarrow E$ be the curvature of \mathcal{K} . If $u, v \in TM$ and $s \in \Gamma(E)$ then the relation between the curvature of \mathcal{K} with the curvature of the connection ∇ is given by $R^E(u, v)s = C(uvs)$, see Paterson [7].

Let $I \subset \mathbf{R}$ be an open interval and consider $a \in I \mapsto J(a)$ a differentiable 1-parameter family of semimartingales in E . Then

$$\begin{aligned} \int \theta D\nabla_a J &= \int \theta \mathcal{K}_\nabla \circ d(\nabla_a J) \\ &= \int \theta \mathcal{K}_\nabla \circ d\mathcal{K}_\nabla \partial_a J \\ &= \int \theta \mathcal{K}_\nabla \mathcal{K}_{\nabla*} \circ d\partial_a J \\ &= \int \theta \mathcal{K}_\nabla \mathcal{K}_{\nabla*} \circ d\partial_a J - \int \theta \mathcal{K}_\nabla \mathcal{K}_{\nabla*} s_E \circ d\partial_a J \\ &\quad + \int \theta \mathcal{K}_\nabla \mathcal{K}_{\nabla*} s_E \circ d\partial_a J \\ &= \int C \circ d\partial_a J + \int \theta \nabla_a D J. \end{aligned}$$

Compare with Arnaudon and Thalmaier [1, Equation 4.13]. An Itô version, as in [1] can be obtained by conversion formulae.

Harmonic sections

Let M be a Riemannian manifold and $\pi : V \rightarrow M$ be a Riemannian vector bundle with a connection ∇ which is compatible with its metric. We denote by E^p the vector bundle $\bigwedge^p T^*M \otimes V$ over M . In this context, we shall consider three differential

geometric operators. The exterior differential operator $d : \Gamma(E^p) \rightarrow \Gamma(E^{p+1})$ is defined by

$$d\sigma(X_1, \dots, X_{p+1}) := (-1)^k (\nabla_{X_k} \sigma)(X_0, \dots, \hat{X}_k, \dots, X_p).$$

The co-differential operator $\delta : \Gamma(E^p) \rightarrow \Gamma(E^{p-1})$ is defined by

$$\delta\sigma(X_1, \dots, X_{p-1}) := -(\nabla_{e_k} \sigma)(e_k, X_1, \dots, X_{p-1}),$$

where $\{e_i\}$ is a local orthonormal frame field. And the Hodge-Laplace operator $\Delta : \Gamma(E^p) \rightarrow \Gamma(E^p)$ is given by

$$\Delta = (d\delta + \delta d).$$

One of the cornerstones of modern geometric analysis is the Weitzenböck formula which states that

$$\Delta\sigma = -\nabla^2\sigma + \Phi(\sigma),$$

for a $\Phi \in \text{End}(E^p)$, see e.g. Eells and Lemaire [4, p.11] or Xin [9, p.21].

Let B_t be a Brownian motion in M and $e_t \in \text{End}(E^p)$ be the solution of

$$D^I e_t = e_t \circ \Phi(B_t) dt.$$

Theorem 3.1 *A section $\sigma \in \Gamma(E^p)$ is harmonic (i.e. $\Delta\sigma = 0$) if and only if for any $\theta \in \Gamma(E^{p*})$*

$$\int \theta D^I \sigma_t$$

is a local martingale, where $\sigma_t = e_t \sigma(B_t)$.

Proof: The result now is consequence of Weitzenböck formula and the following

Lemma 3.1 *Consider $\sigma \in \Gamma(E^p)$, $\theta \in \Gamma(E^{p*})$ and a semimartingale x_t in M . Given $V \in \text{End}(E^p)$, let $e_t \in \text{End}(E^p)$ be the solution of*

$$D^I e_t = e_t \circ V(x_t) g(dx_t, dx_t).$$

Write $\sigma_t = e_t \sigma(x_t)$. Then

$$\int \theta D^I \sigma_t = \int (\theta \circ \nabla \sigma) d^{\nabla^M} x_t + \int (\theta \circ e_t) \left(\frac{1}{2} \nabla^2 + V(\sigma(x_t))g \right) \sigma(dx_t, dx_t).$$

Proof: By covariant Itô-Stratonovich conversion formula, Equation (2), we have that

$$\begin{aligned} \int \theta D^I \sigma_t &= \int \theta D^I e_t(\sigma(x_t)) + \int (\theta \circ e_t) D^I(\sigma(x_t)) \\ &= \int (\theta \circ e_t) V(\sigma(x_t)) g(dx_t, dx_t) + \int (\theta \circ e_t) D^S(\sigma(x_t)) \\ &\quad + \frac{1}{2} \int \nabla(\theta \circ e_t)(dx_t, D\sigma(x_t)). \end{aligned} \tag{7}$$

Now, by usual Itô-Stratonovich conversion formula:

$$\begin{aligned}
\int (\theta \circ e_t) D^S(\sigma(x_t)) &= \int (\theta \circ e_t) \mathcal{K}_\nabla \sigma_* dx_t \\
&= \int (\theta \circ e_t) \nabla \sigma d^{\nabla^M} x_t \\
&\quad - \frac{1}{2} \int \nabla^M(\theta \circ e_t \circ \nabla \sigma)(dx_t, dx_t).
\end{aligned} \tag{8}$$

We have that

$$\int \nabla(\theta \circ e_t) (dx_t, D\sigma(x_t)) = \int \nabla(\theta \circ e_t) \circ (I \otimes \nabla \sigma) (dx_t, dx_t) \tag{9}$$

substituting (8) and (9) in (7) one finds:

$$\begin{aligned}
\int \theta D^I \sigma_t &= \int (\theta \circ e_t) V(\sigma(x_t)) g(dx_t, dx_t) + \int (\theta \circ e_t) \nabla \sigma d^{\nabla^M} x_t \\
&\quad - \frac{1}{2} \int \nabla^M(\theta \circ e_t \circ \nabla \sigma)(dx_t, dx_t) \\
&\quad + \frac{1}{2} \int \nabla(\theta \circ e_t) \circ (I \otimes \nabla \sigma) (dx_t, dx_t).
\end{aligned}$$

The result follows using that for all $\theta \in \Gamma(E^*)$,

$$\nabla \theta \circ (I \otimes \nabla \sigma) - \nabla^M(\theta \circ \nabla \sigma) = \theta(\nabla^2 \sigma).$$

□

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